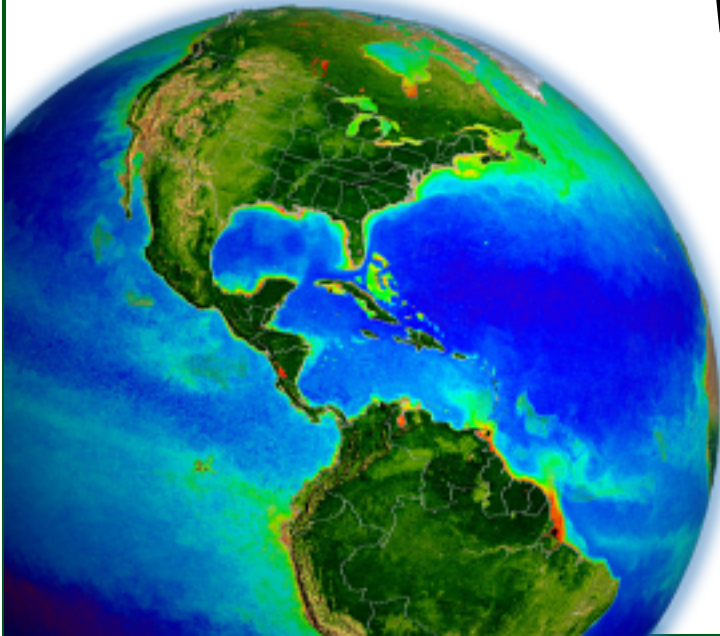


Atmospheric Correction Discussion

*where we are today, and
where we might be in 2.5 years*




B. Franz, PACE ST Meeting,
College Park, MD, January 2014

The goal of atmospheric correction (AC) is to convert observed top-of-atmosphere spectral radiance to water-leaving reflectance (R_{rs}) over the NUV-VIS spectral regime?

current NASA atmospheric correction approach

$$L_t = \left(L_r + [L_a + L_{ra}] + t_{dv} L_f + T_s T_v L_g + t_{dv} L_w \right) t_{gv} t_{gs} f_p$$

Observed TOA
Rayleigh
Aerosol + Ray-Aer
foam & whitecaps
Sun glint
water-leaving

$$R_{rs} = \frac{L_w}{F_0 \cos(\theta_s) t_{ds} f_s} f_b f_\lambda$$


current NASA atmospheric correction approach

instrument polarization correction

$$L_t = \left(L_r + [L_a + L_{ra}] + t_{dv} L_f + T_s T_v L_g + t_{dv} L_w \right) t_{gv} t_{gs} f_p$$

$$R_{rs} = \frac{L_w}{F_0 \cos(\theta_s) t_{ds} f_s f_b f_\lambda}$$

solar constant (irradiance, Thuillier 2003) &
an adjustment for the Earth-Sun distance

bidirectional reflectance
correction

correction for out-of-band
response

current NASA atmospheric correction approach

transmittance of gases (g) in direction of
Sun (s) or satellite (v)

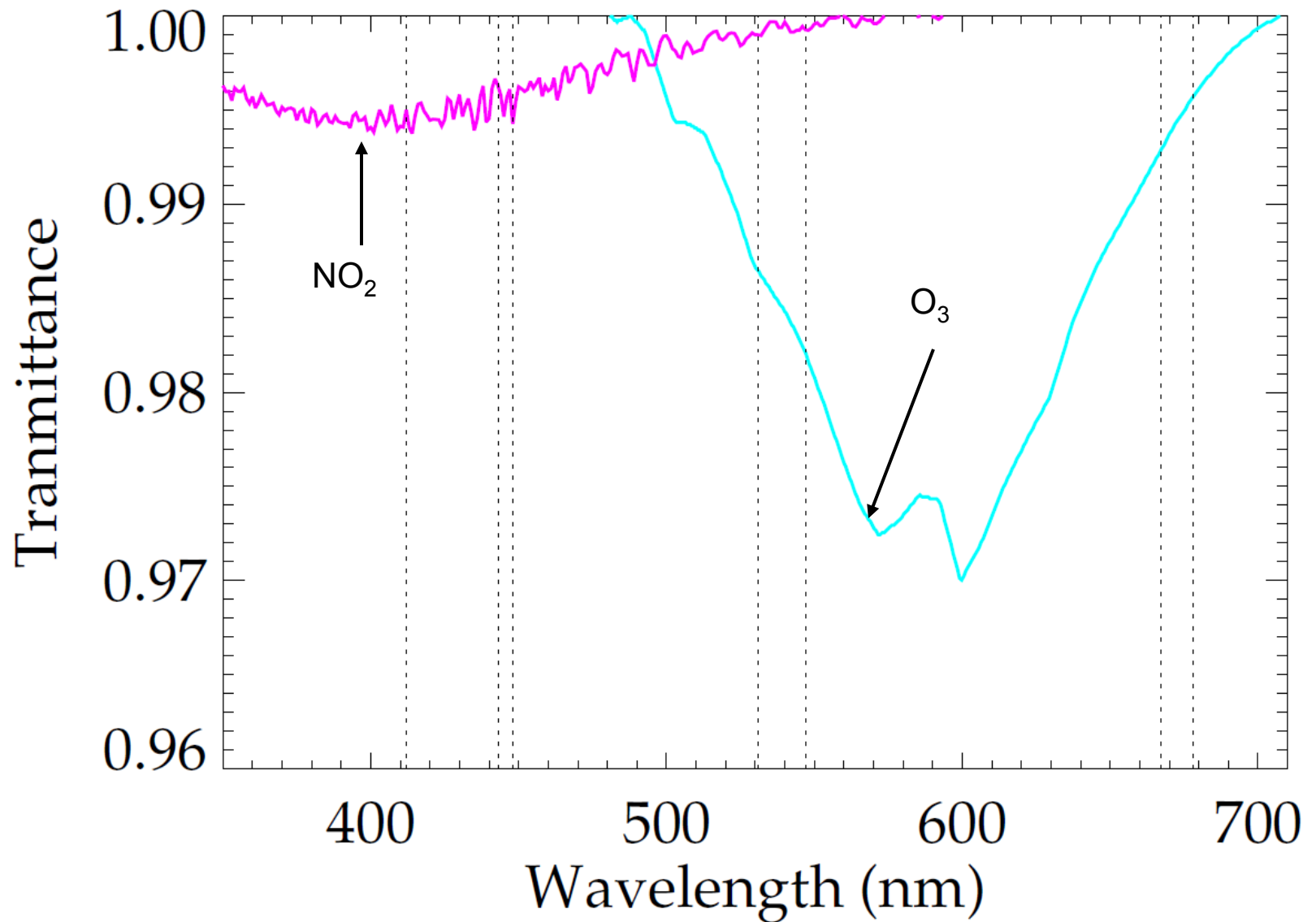
$$L_t = \left(L_r + [L_a + L_{ra}] + t_{dv} L_f + T_s T_v L_g + t_{dv} L_w \right) t_{gv} t_{gs} f_p$$

$$R_{rs} = \frac{L_w}{F_0 \cos(\theta_s) t_{ds} f_s} f_b f_\lambda$$

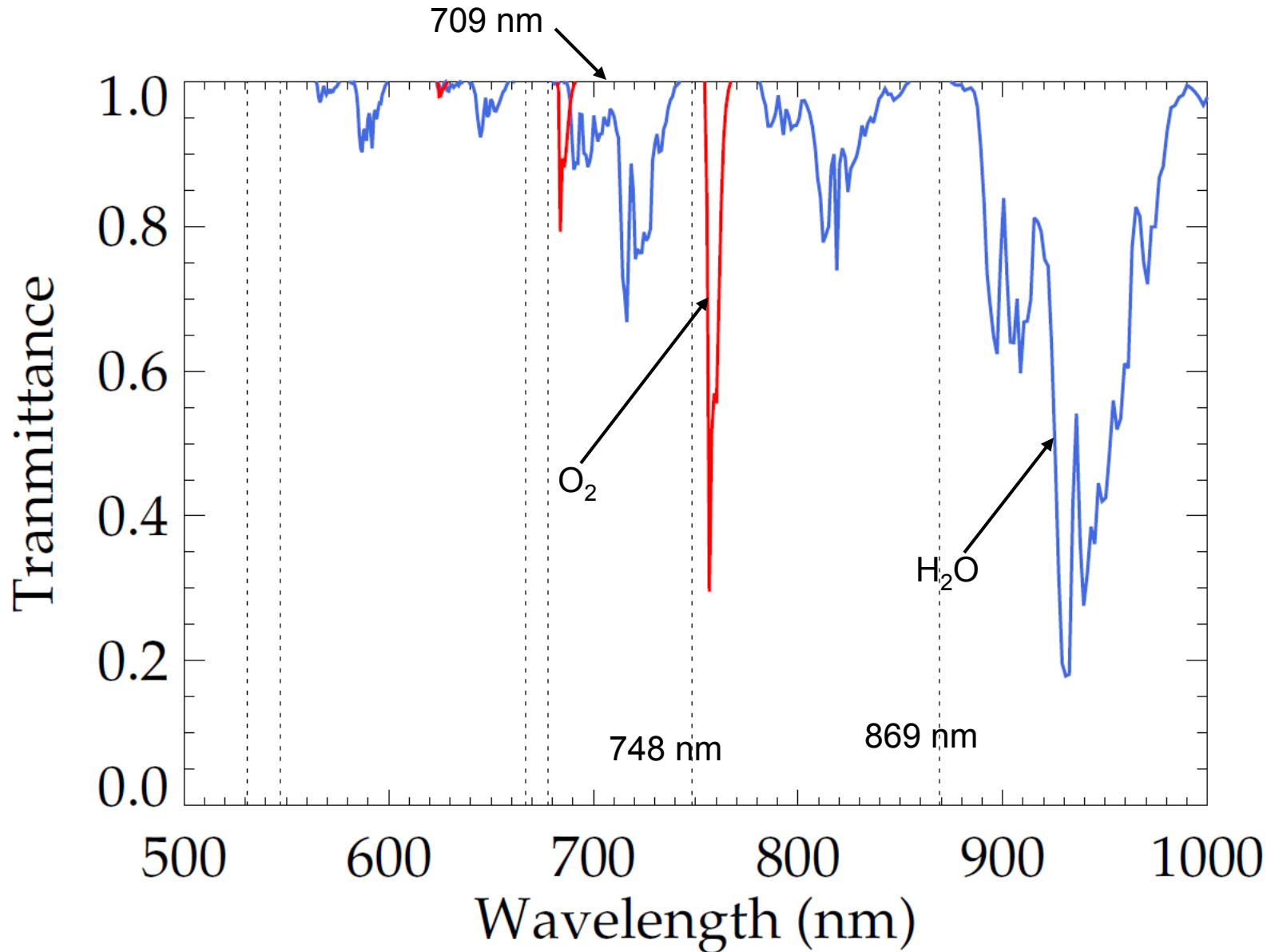
Rayleigh / aerosol diffuse
transmittance (d) in direction of Sun
(s) or satellite (v)

note: no coupling between gaseous absorption and scattering terms

gaseous transmittance (O₃ & NO₂)



gases not directly considered (O_2 & H_2O)



current NASA atmospheric correction approach

$$L_t = \left(L_r + [L_a + L_{ra}] + t_{dv} L_f + T_s T_v L_g + t_{dv} L_w \right) t_{gv} t_{gs} f_p$$

Observed TOA
Rayleigh
Aerosol + Ray-Aer
foam & whitecaps
Sun glint
water-leaving

$$R_{rs} = \frac{L_w}{F_0 \cos(\theta_s) t_{ds} f_s} f_b f_\lambda$$

foam & whitecaps

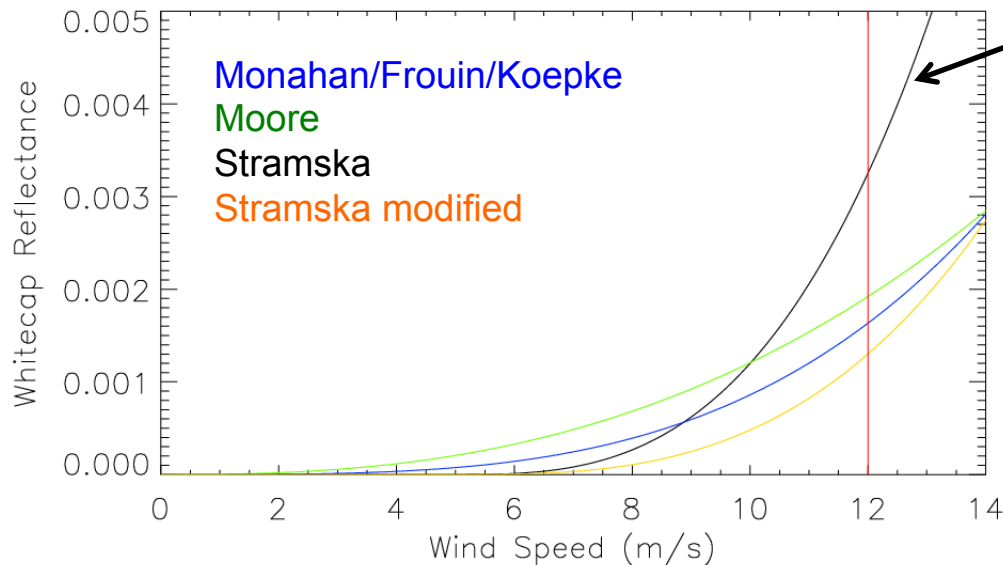
$$L_f(\lambda) = \rho_f(\lambda) [a (U_{10} + b)^3] F_0 \cos(\theta_0) / \pi$$

$\rho_f(\lambda)$ = effective whitecap reflectance

22% from Koepke 1984

NIR spectral dependence from Frouin 1999

U_{10} = wind speed at 10-meters (≤ 12 m/s)



Stramska & Petelski, JGR, 2003

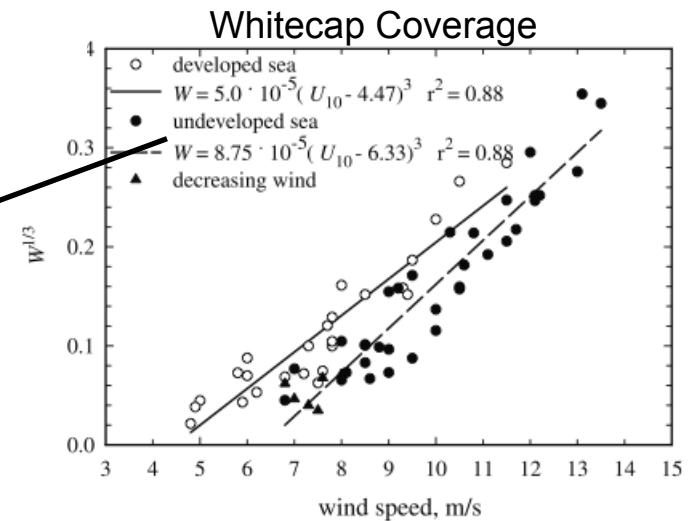


Figure 8. Oceanic whitecap coverage as a function of wind speed. Different symbols are used for the developed wave field, the undeveloped wave field, and the decreasing wind speed. See text for details.

Sun glint

$$L_g = F_0 L_{GN}$$

$$T_s T_v = \exp \left[-(\tau_r + \tau_a) \left(\frac{1}{\cos(\theta_0)} + \frac{1}{\cos(\theta)} \right) \right]$$

L_{GN} from Cox and Munk (1954)

glint radiance normalized to no atmosphere & $F_0 = 1$

statistical function of windspeed

flagged as high glint if $L_{GN} > 0.005$ and masked in Level-3

Actually using a two step iteration since we don't know τ_a :

- (1) $[L_t, \tau_a', W] \rightarrow L_t^{(1)} = L_t - TL_g \rightarrow \tau_a^{(1)}$
- (2) $[L_t^{(1)}, \tau_a^{(1)}, W] \rightarrow L_t^{(2)} = L_t^{(1)} - TL_g \rightarrow \tau_a^{(2)}$

with initial guess of $\tau_a' \sim 0.1$

Wang & Bailey 2001

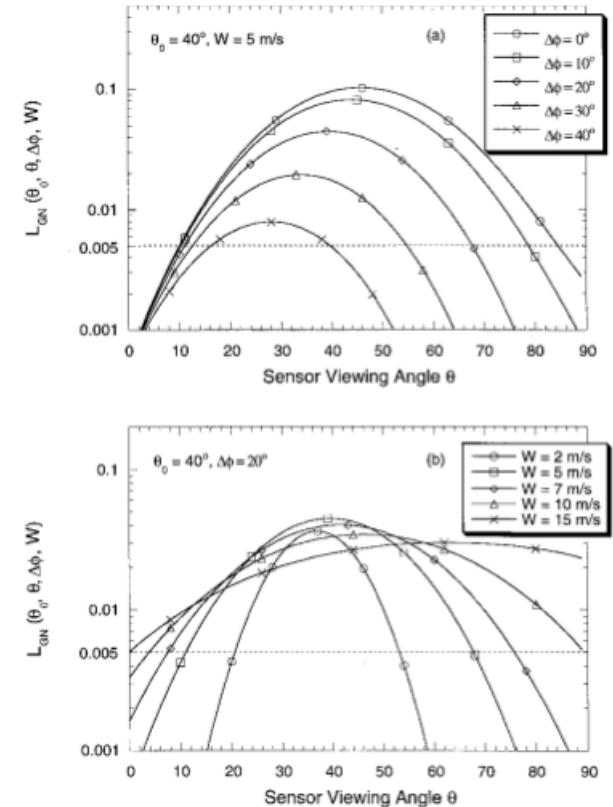


Fig. 1. Normalized sun glint radiance L_{GN} as a function of the sensor-viewing angle (solar zenith angle, 40°) and for (a) various relative azimuthal angles with surface wind speed of 5 m/s and (b) various surface wind speeds with a relative azimuthal angle of 20° .

molecular (Rayleigh) scattering

Using pre-computed look-up tables from Ahmad & Fraser vector radiative transfer simulations for wind-roughened ocean surface.

Input bandpass-integrated Rayleigh optical thickness was computed using the model of Bodaine et al. (1999).

Rayleigh radiances (with polarization, I, Q, U) are retrieved from look up tables and adjusted given:

- solar & satellite viewing geometries (θ_0 , θ , $\Delta\phi$)
- windspeed (a proxy for surface roughness)
- atmospheric pressure (to adjust for change in optical thickness, τ_r)

Rayleigh reflectance calculable to ~0.2% (bias)

based on RT intercomparisons, before vicarious calibration

current NASA atmospheric correction approach

$$L_t = \left(L_r + \underbrace{[L_a + L_{ra}]}_{\text{aerosol + Ray-aer}} + t_{dv} L_f + T_s T_v L_g + t_{dv} L_w \right) t_{gv} t_{gs} f_p$$

Observed TOA
Rayleigh
aerosol + Ray-aer
foam & whitecaps
Sun glint
water-leaving

$$R_{rs} = \frac{L_w}{F_0 \cos(\theta_s) t_{ds} f_s} f_b f_\lambda$$

←

aerosol contribution (basic concept)

assume $L_w(\lambda) = 0$ at two NIR (or SWIR) bands, **or that it can be estimated with sufficient accuracy.**

retrieve aerosol reflectance in each NIR band as

$$[L_a + L_{ra}] = \frac{L_t}{t_{gv} t_{gs} f_p} - \left(L_r + t_{dv} L_f + T_s T_v L_g + t_{dv} \cancel{L_w} \right)$$

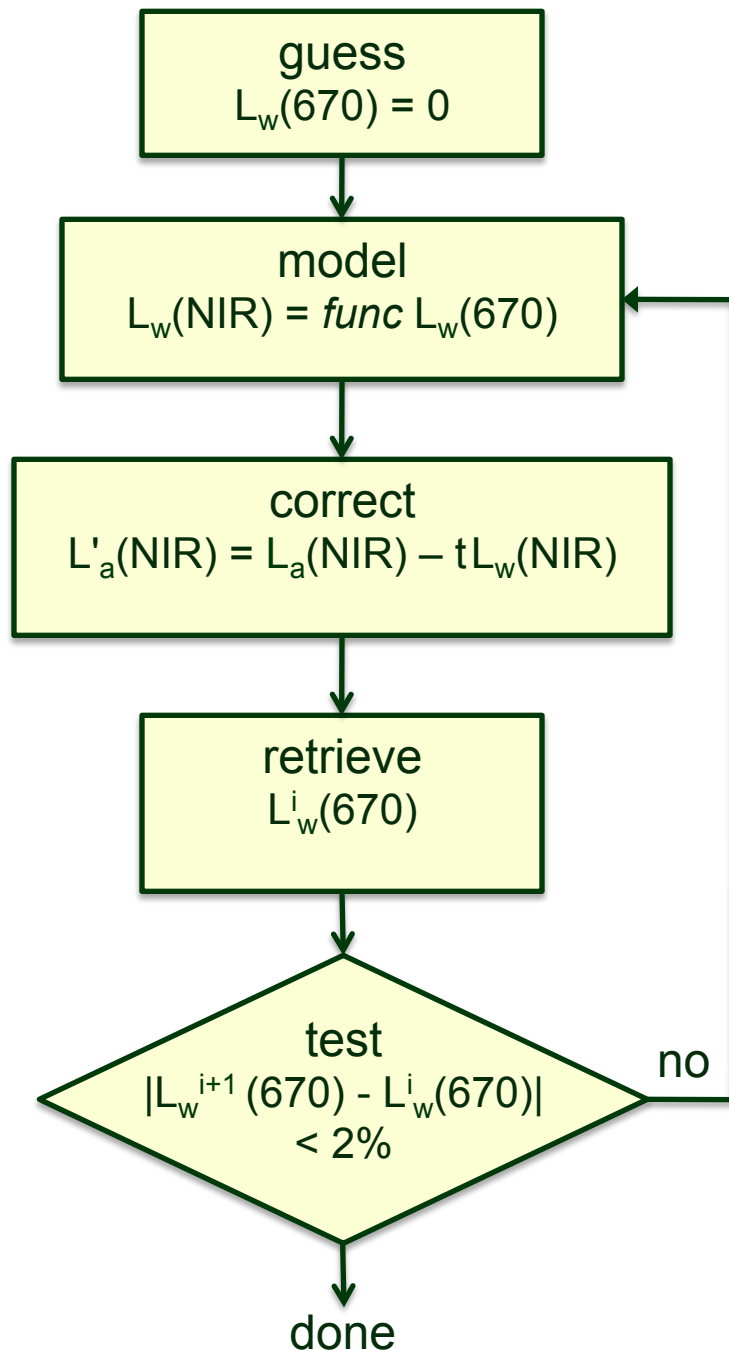
known
↑

$$\rho_a = [L_a + L_{ra}] \frac{\pi}{F_0 \cos(\theta_0)}$$

use spectral dependence of retrieved NIR aerosol reflectance (ε) to select the most appropriate aerosol model from a suite of pre-computed models

use NIR aerosol reflectance and selected aerosol model to extrapolate aerosol reflectance to visible wavelengths

we estimate $L_w(\text{NIR})$ using a bio-optical model



1) convert $L_w(670)$ to $b_b/(a+b_b)$
via Morel f/Q and retrieved Chl_a

2) estimate $a(670) = a_w(670) + a_{pg}(670)$
via NOMAD empirical relationship

$$a(670) = e^{(\ln(C_a) * 0.9389 - 3.7589)} + a_w(670)$$

3) estimate $b_{bp}(\text{NIR}) = b_{bp}(670) (\lambda/670)^\eta$
via Lee et al. 2002

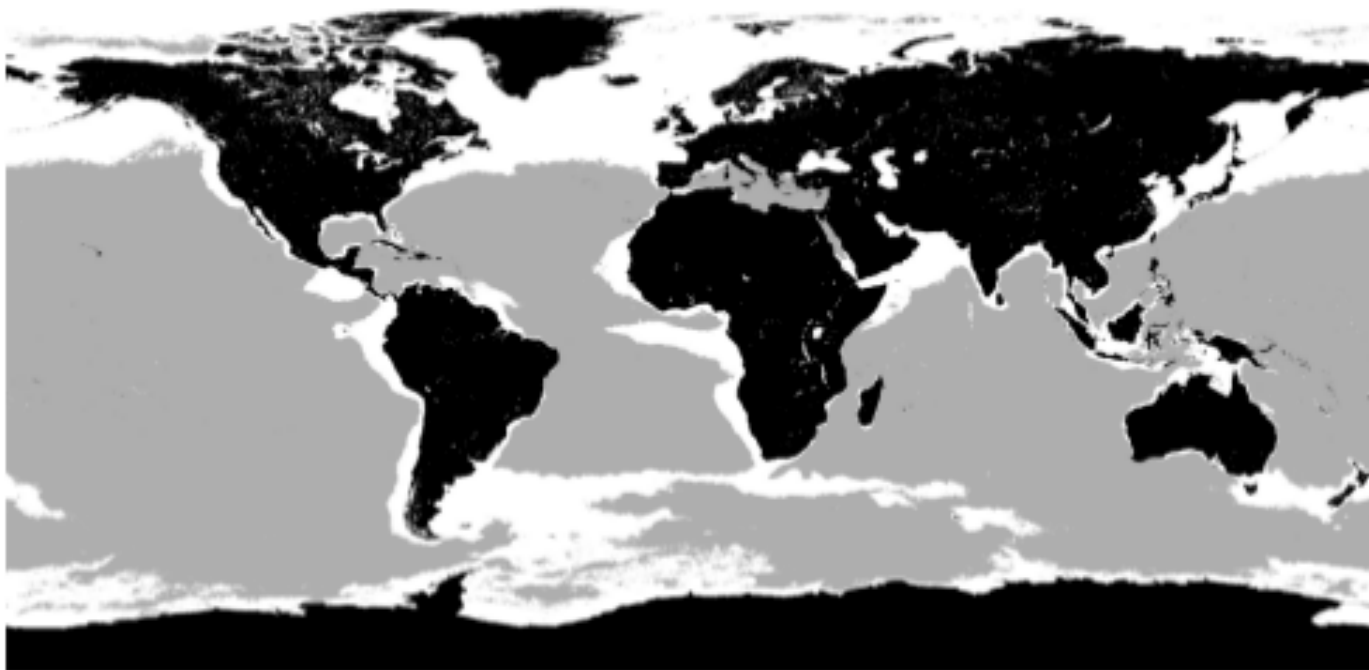
$$\eta = 2.0 * \left[1. - 1.2 * e^{(-0.9 * R_{rs}(443)/R_{rs}(555))} \right]$$

4) assume $a(\text{NIR}) = a_w(\text{NIR})$

5) estimate $L_w(\text{NIR})$ from $b_b/(a+b_b)$
via Morel f/Q and retrieved Chl_a

locations (white) where $L_w(\text{NIR})$ is significant

locations of application of bio-optical model



not applied when $\text{Chl} < 0.3 \text{ mg m}^{-3}$

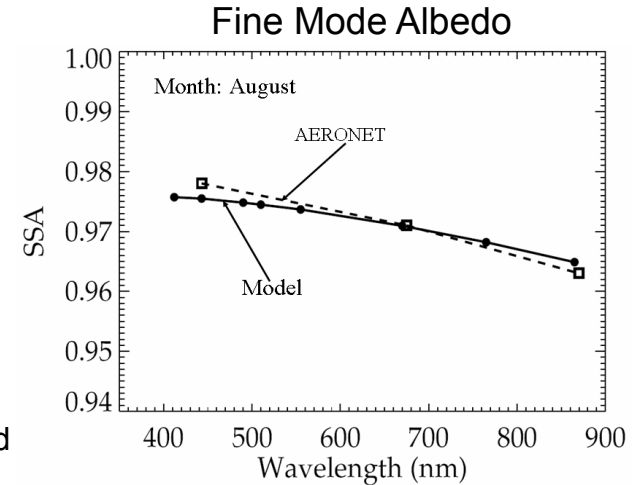
weighted application when $0.3 < \text{Chl} < 0.7 \text{ mg m}^{-3}$

fully applied when $\text{Chl} > 0.7 \text{ mg m}^{-3}$

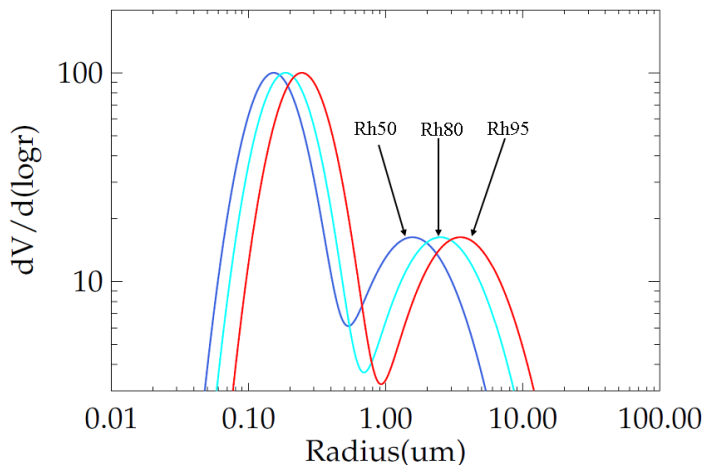
Bailey et al., Optics
Express, 2010

aerosol model tables

- vector RT code (Ahmad-Fraser)
- based on AERONET size distributions & albedos
- 80 models (10 size fractions within 8 humidities)
 - 100% coarse mode to 95% fine mode
 - non- or weakly absorbing
- LUT: extinction, albedo, phase function, ss->ms, t_d
 - function of path geometry (or scattering angle)
- model selection discriminated by relative humidity



Typical Size Distributions



Mean AERONET Fine & Coarse Modal Radii

Rh	r_{vf}	σ_f	r_{vc}	σ_c	r_{vf}/r_{ovf}	r_{vc}/r_{ovc}
0.30	0.150	0.437	2.441	0.672	1.006	1.009
0.50	0.152	0.437	2.477	0.672	1.019	1.024
0.70	0.158	0.437	2.927	0.672	1.063	1.210
0.75	0.167	0.437	3.481	0.672	1.118	1.439
0.80	0.187	0.437	3.966	0.672	1.255	1.639
0.85	0.204	0.437	4.243	0.672	1.371	1.753
0.90	0.221	0.437	4.638	0.672	1.486	1.917
0.95	0.246	0.437	5.549	0.672	1.648	2.293

aerosol model selection & application

select the two sets of 10 models (10 size fractions) with relative humidity (RH) that bound the RH of the observation.

find the two models that bound the observed epsilon within each RH model family.

$$\epsilon^{obs}(748, 869) = \frac{\rho_a(748)}{\rho_a(869)} \rightarrow \epsilon^{mod}(748, 869)$$

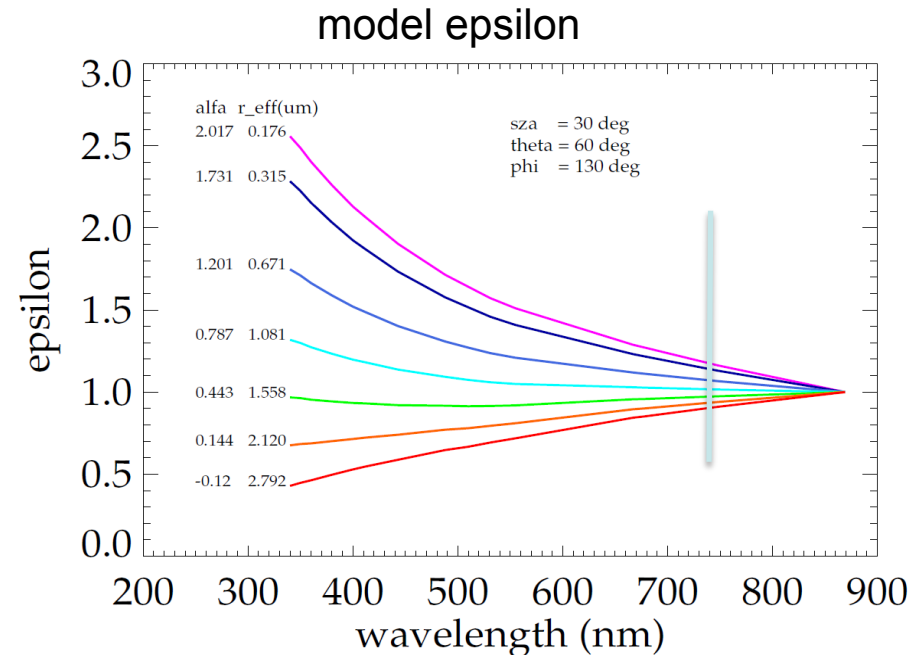
use model epsilon to extrapolate to visible.

$$\rho_a(\lambda) = \rho_a(869) \epsilon^{mod}(\lambda, 869)$$

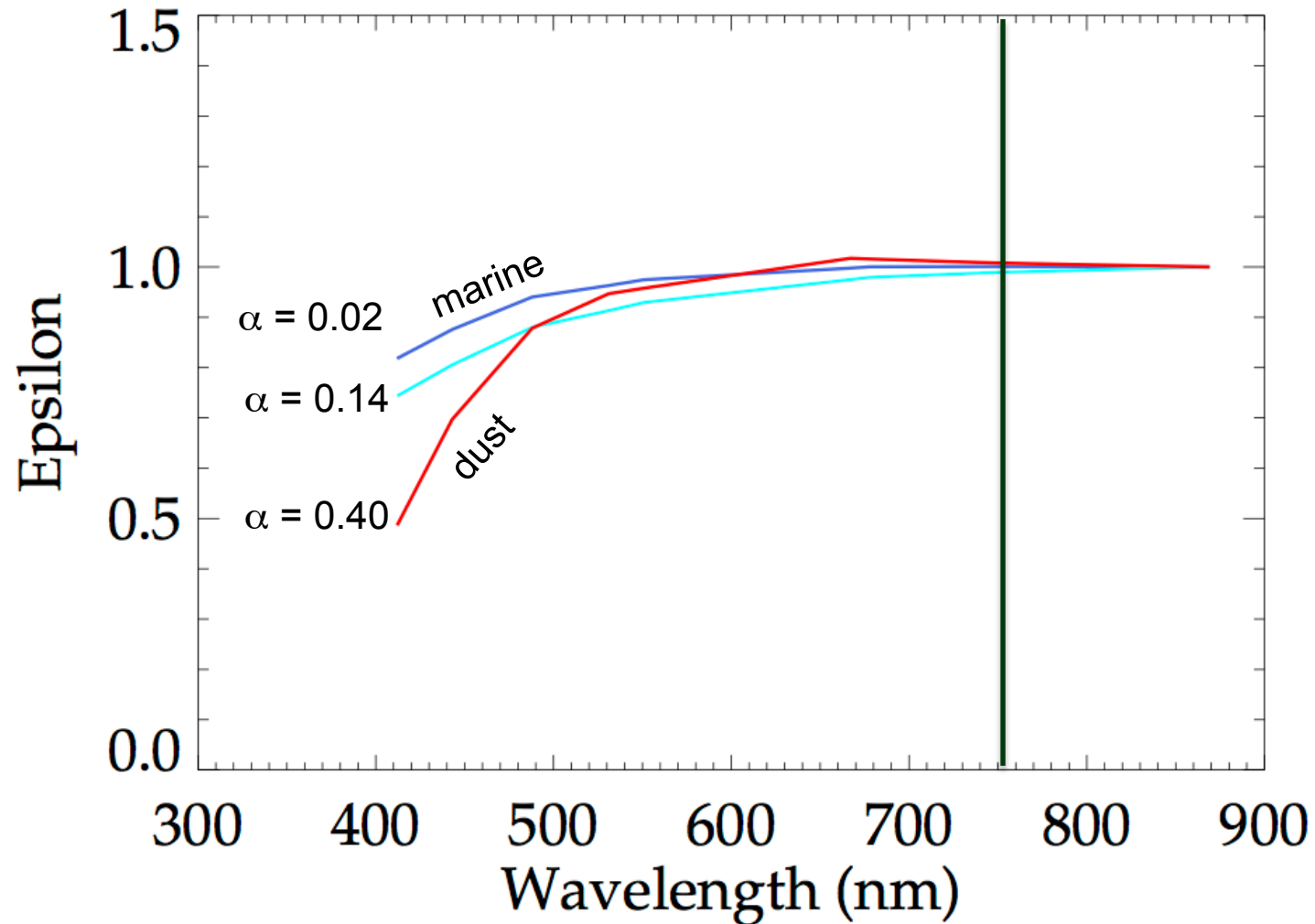
compute weighted average, $\bar{\rho}_a$, between models within each RH family, and then again between bounding RH solutions.

$$[L_a + L_{ra}] = \bar{\rho}_a(\lambda) \frac{F_0 \cos(\theta_0)}{\pi}$$

**actually done in single scattering space and transformed to multi-scattering*

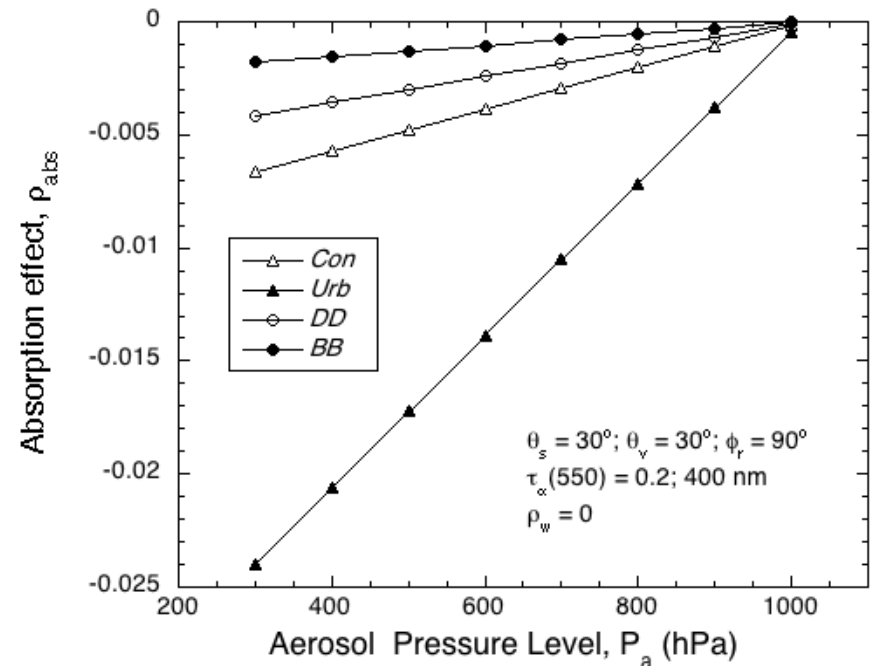
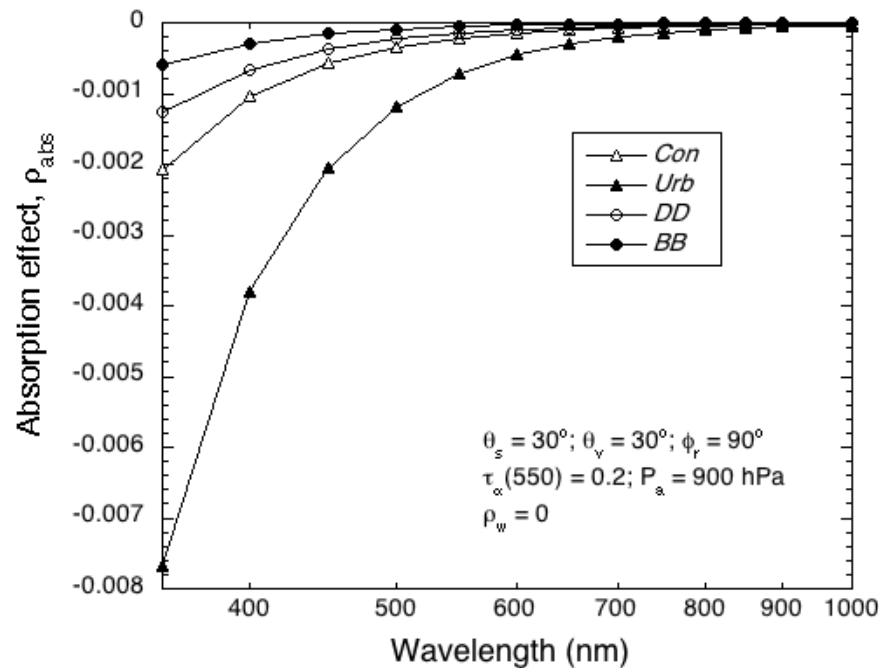


can't distinguish absorbing aerosols using NIR spectral dependence alone



Measurements in two spectral bands in the red, near-infrared, and/or short-wave infrared are not sufficient, in the general case, to determine the perturbing effects of the atmosphere and surface in the visible.

No sensitivity to aerosol absorption, information about aerosol altitude needed.



Absorption effect as a function of wavelength (top left), air mass (bottom left), and aerosol pressure level (top right) for continental, urban, desert dust, and biomass burning aerosol models. The effect increases in magnitude with decreasing wavelength, decreasing aerosol pressure level, and increasing air mass.

note: NIR aerosol reflectance is an error bucket

- any error in subtraction of Rayleigh, whitecaps, or glint, or any signal that is not identified and subtracted (e.g., thin cirrus, cloud edges, straylight, atmospheric adjacency) will add to the aerosol reflectance in the NIR and be extrapolated to the visible via the aerosol model.
- many of these sources are approximately white, so likely the effect is to flatten the retrieved spectral dependence proportionate to the residual signal.
- both the aerosol concentration and aerosol type will be altered by such errors, and thus the aerosol properties will be inaccurate, but the reflectance that is subtracted “may” be approximately correct.

current NASA atmospheric correction approach

$$L_t = \left(L_r + [L_a + L_{ra}] + t_{dv} L_f + T_s T_v L_g + t_{dv} L_w \right) t_{gv} t_{gs} f_p$$

Observed TOA
Rayleigh
aerosol + Ray-aer
foam & whitecaps
Sun glint
water-leaving

$$R_{rs} = \frac{L_w}{F_0 \cos(\theta_s) t_{ds} f_s} f_b f_\lambda$$

bidirectional reflectance
correction

brdf correction

to account for shape of sub-surface light-field due to position of the Sun and optical properties of the water column.

based on pre-computed look-up tables from hydrolight simulations of Morel et al. 2002, Appl. Opt.

given radiant path geometry $(\theta_0, \theta, \Delta\varphi)$, windspeed (w) and **Chl**

$$\text{Chl} = f(R'_{rs}(\lambda))$$

$$R'_{rs} = \frac{L_w}{F_0 \cos(\theta_s) t_{ds} f_s} f_\lambda$$

$$\theta_0=0, \theta=0, \Delta\varphi=0$$

$$f_b(\lambda, \theta_0, \theta, \Delta\varphi, \text{Chl}, w) = (\mathcal{R}_0 f_0 / Q_0) / (\mathcal{R} f / Q)$$

f/Q relates subsurface irradiance reflectance to radiance reflectance
 \mathcal{R} includes all reflection/refraction effects of the air-sea interface

$$R_{rs}(\lambda) = R'_{rs}(\lambda) f_b(\lambda, \theta_0, \theta, \Delta\varphi, \text{Chl}, w)$$

$$\text{Chl} = f(R_{rs}(\lambda))$$

iteration

cloud masking

- over ocean, we attempt to process all pixels with valid radiometry and navigation, that are not classified as cloud.
- the standard cloud mask is just a threshold on surface + aerosol reflectance (excluding glint) at ~865nm.

$$[L_a + L_{ra}] + t_{dv}L_f + t_{dv}L_w = \frac{L_t}{t_{gv}t_{gs}} - (L_r + T_sT_vL_g)$$

$$\rho_c = \left[\frac{L_t}{t_{gv}t_{gs}} - (L_r + T_sT_vL_g) \right] \frac{\pi}{F_0 \cos(\theta_0)}$$

$$\rho_c > 0.022 = \text{cloud}$$

- note: any bright signal in NIR will be classified as cloud (e.g., uncorrected glint, ice, high suspended sediment loads, bottom reflectance, high concentration of coccolithophores).

summary of current atmospheric correction

- correction for atmospheric absorption
 - O₃, NO₂ (O₂ & H₂O only for out-of-band responsivity corrections)
 - using coincident O₃ and climatological NO₂
- surface and sub-surface corrections (windspeed dependent)
 - whitecaps, model of Stramska and Petelski (2003)
 - Sun glint, statistical model based on Cox & Munk
 - brdf (fresnel + Morel f/Q)
- subtraction of radiance scattered by air molecules
 - pre-computed Rayleigh scattering look-up tables w/polarization
 - function of geometry and windspeed, also uses surface pressure
- subtraction of radiance scattered by aerosols + Rayleigh-aerosol
 - aerosol contribution derived from reflectance in 2 NIR bands + models
 - aerosol models derived from AERONET measurements (non- or weakly-absorbing marine and coastal aerosols)
 - depends on ancillary relative humidity

summary of ancillary data requirements

ancillary data

atmospheric pressure
relative humidity
wind speed
ozone
NO₂
water vapor

ancillary source

NCEP
NCEP
NCEP
OMI/TOMS
Sciamachy/OMI/GOME
NCEP

uses

Rayleigh
aerosol models
white caps, Sun glint, Rayleigh
transmittance
transmittance
out-of-band correction

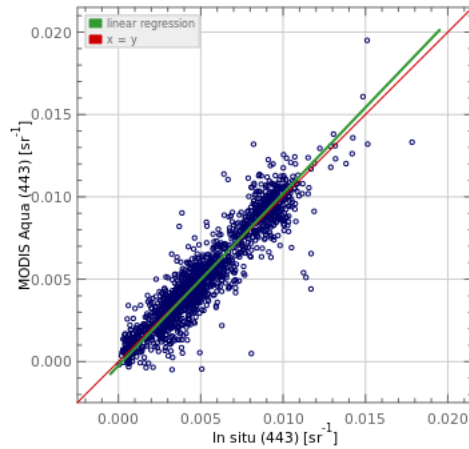
tables, coefficients

solar irradiances
aerosol models
Rayleigh reflectance
ozone absorption coef
NO₂ absorption coef
f/Q (bidirectional reflectance distributions)

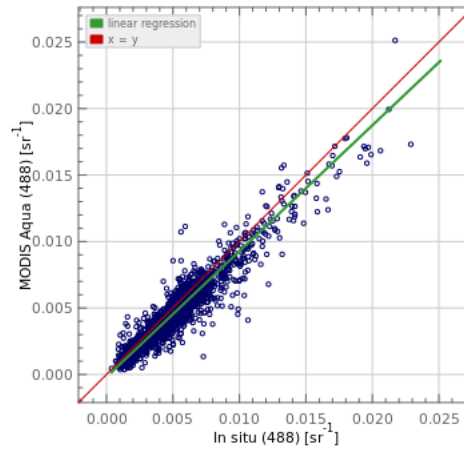
how well does it work?

MODISA Rrs Validation (SeaBASS + AERONET-OC)

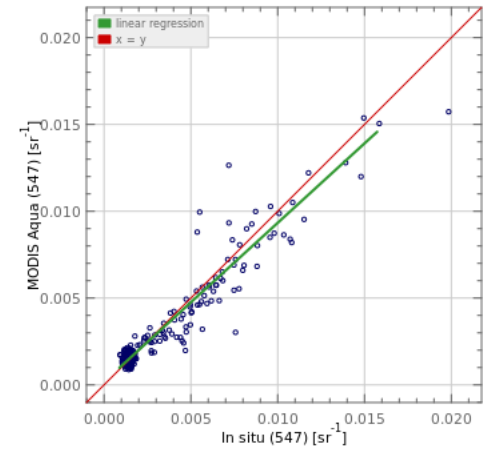
Rrs(443)



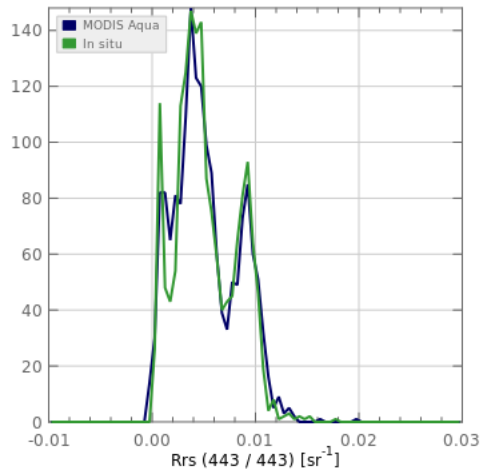
Rrs(488)



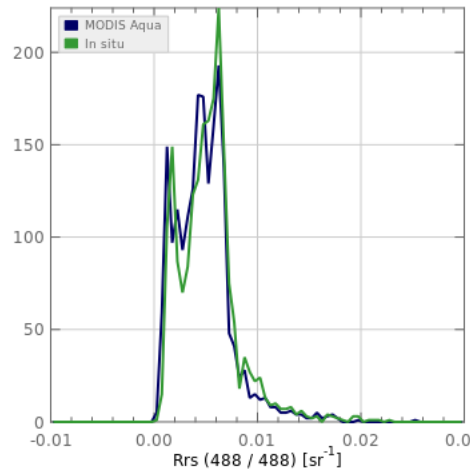
Rrs(547)



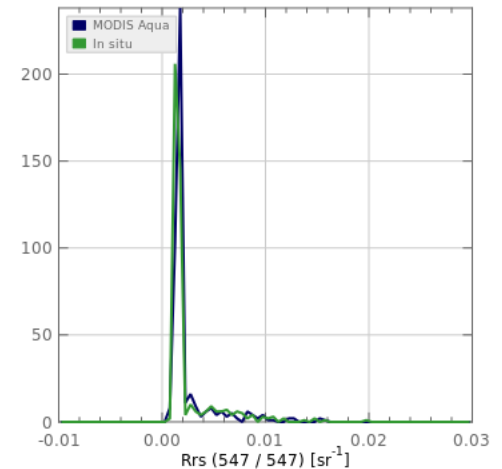
Frequency Distribution



Frequency Distribution



Frequency Distribution



Mean APD 12-13%, Mean Bias < 10%, $R^2 > 0.9$

how well does it work?

MODISA Rrs Validation (SeaBASS + AERONET-OC)

Product Name	MODIS Aqua Range	In situ Range	#	Best Fit Slope	Best Fit Intercept	R ²	Median Ratio	Abs % Difference	RMSE
Rrs412	-0.00411, 0.01820	0.00000, 0.01964	1945	1.03539	-0.00065	0.90481	0.90307	22.21457	0.00147
Rrs443	-0.00065, 0.01950	0.00005, 0.01783	1774	1.04628	-0.00026	0.88967	1.00894	12.06771	0.00109
Rrs488	0.00033, 0.02513	0.00039, 0.02289	2127	0.94853	-0.00021	0.89894	0.91509	12.00520	0.00106
Rrs531	0.00092, 0.01682	0.00130, 0.02110	639	0.87525	0.00017	0.91346	0.97562	11.98040	0.00096
Rrs547	0.00088, 0.01590	0.00091, 0.01984	469	0.91611	0.00018	0.92442	1.04480	13.38668	0.00072
Rrs667	-0.00016, 0.01186	0.00002, 0.01100	709	0.98687	-0.00002	0.91982	0.94565	37.48856	0.00017
Rrs678	-0.00015, 0.00283	0.00004, 0.00295	373	0.94854	-0.00000	0.89380	1.00161	32.16394	0.00008

The linear regression algorithm has been changed to reduced major axis.

Rrs uncertainty goals (PACE SDT)

open ocean, clear-water, marine aerosols

$[\rho_w(\lambda)]_N$ $\lambda=400-710\text{nm}$, maximum of 0.001 or 5% (VIS)

$[\rho_w(\lambda)]_N$ $\lambda=350-400\text{nm}$, maximum of 0.002 or 10% (NUV)

in terms of $R_{rs}(\lambda) = [\rho_w(\lambda)]_N/\pi$, that is:

$$\Delta R_{rs}(\lambda) = 3\text{e-}4 \text{ (sr}^{-1}\text{, VIS)}$$

$$\Delta R_{rs}(\lambda) = 6\text{e-}4 \text{ (sr}^{-1}\text{, NUV)}$$

how well does it work?

MODISA Rrs Validation (SeaBASS + AERONET-OC)

Product Name	MODIS Aqua Range	In situ Range	#	Best Fit Slope	Best Fit Intercept	R ²	Median Ratio	Abs % Difference	RMSE
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Rrs667	-0.00016, 0.01186	0.00002, 0.01100	709	0.98687	-0.00002	0.91982	0.94565	37.48856	0.00017
Rrs678	-0.00015, 0.00283	0.00004, 0.00295	373	0.94854	-0.00000	0.89380	1.00161	32.16394	0.00008

The linear regression algorithm has been changed to reduced major axis.

PACE SDT Goal for Rrs(VIS)

$$\Delta R_{rs}(VIS) = 3e-4 \text{ sr}^{-1} \text{ or } 5\%$$

Current Approach

$$\Delta R_{rs}(VIS) \sim 1e-3 \text{ sr}^{-1} \text{ or } 12\% \text{ (22\% 412)}$$

goal is factor of 3 reduction ... seems achievable!

questions on heritage algorithm?

where do we go from here?

I see two complementary paths:

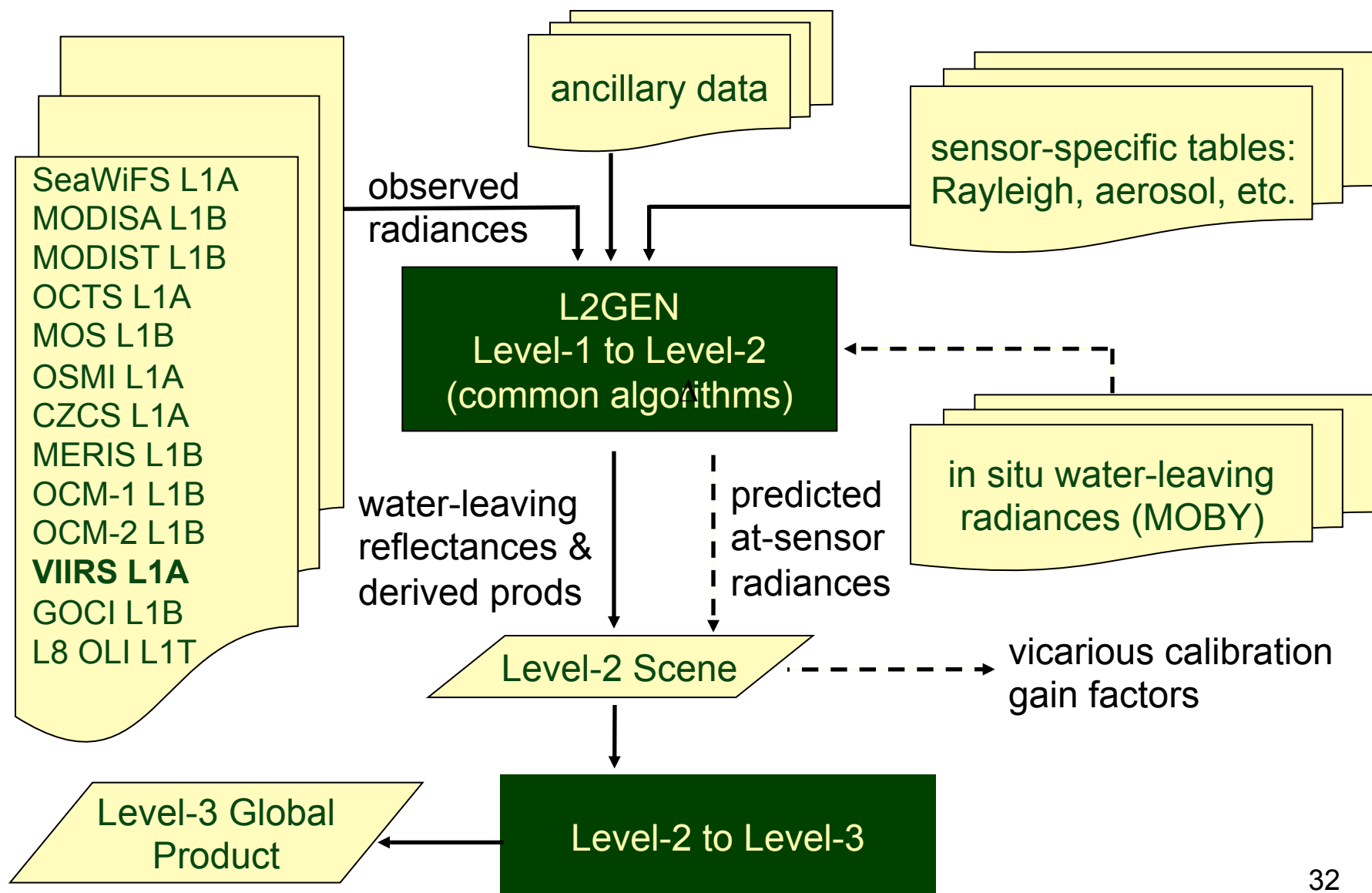
1. adapt and improve the heritage algorithm to support PACE SDT notional instruments (*Franz, Gao*)
2. develop a completely different approach (i.e., simultaneous retrieval of atmosphere and R_{rs} /IOPs (*Chowdhary, Frouin*))

with both paths potentially benefiting from work within the Team, e.g.:

- absorbing aerosol identification (NUV) and modeling
- aerosol height (O2 A-band)
- absorbing gas corrections (H2O)
- whitecaps
- cloud flagging or cloud corrections

Implementation

NASA Standard Processing Code



I2gen already supports many different atmospheric correction options and algorithms

- aerosol model selection
 - specifiable band pairs, including NIR-SWIR switching
 - specifiable model suites, with or without RH stratification
 - model selection in single or multi-scattering space
 - various methods of NIR Lw estimation (*Bailey et al. 2010, Ruddick et al. 2000*)
- simultaneous atmospheric correction and IOP retrieval
 - Spectral Optimization Algorithm (*Chomko & Gordon 1998, Kuchinke et al. 2009*)
 - Spectral Matching Algorithm (*Gordon et al. 1997, Moulin et al. 2001*)
- modular components for whitecaps, glint, gas transmittances, etc.

where do we go from here?

I see two complementary paths:

1. adapt and improve the heritage algorithm to support PACE SDT notional instruments (*Franz, Gao*)
2. develop a completely different approach (i.e., simultaneous retrieval of atmosphere and R_{rs} /IOPs (*Chowhdary, Frouin*))

with both paths potentially benefiting from work within the Team, e.g.:

- absorbing aerosol identification (NUV) and modeling
- aerosol height (O2 A-band)
- absorbing gas corrections (H2O)
- whitecaps
- cloud flagging or cloud corrections

using I2gen as the implementation framework

discussion

Is it reasonable to expect a working implementation of the heritage algorithm for a PACE-like radiometer within 2.5 years?

yes

- 1. modify l2gen to support hyperspectral (in progress)*
- 2. add simple water-vapor correction (working with Gao)*
- 3. start testing on HICO, AVIRIS*
- 4. start developing/testing algorithm enhancements (e.g.)*
 - aerosol selection in multi-scattering space*
 - use of more than two bands (minimization over NIR-SWIR atm. windows), adaptive NIR-SWIR band-set selection*
 - incorporating other developments within ST (abs. aerosol detection, whitecaps, cirrus, etc.)*

Is it reasonable to expect a working implementation of at least one alternative algorithm (e.g., ACROSS) within 2.5 years?

How will we evaluate algorithm performance/behavior?

- simulated data: controlled experiment, answer is known, but does not test real-world conditions, may favor one algorithm where forward and inverse models are common*
- aircraft/spacecraft data + co-incident field measurements: no perfect match to PACE notional sensors (OCI, OCI+, etc.), sensor-specific calibration issues may (will) confound results, uncertainty in field measurements*

What are the likely advancements we can demonstrate in 2.5 years?

- retrieval of R_{rs} in NUV-VIS for open ocean, marine aerosols?*
- absorbing aerosol detection? correction? accounting for aerosol height?*
- can we improve identification and correction for non or weakly-absorbing aerosols?*
- improved cloud detection (cloud correction)?*
- improved glint correction? whitecap correction?*
- improved brdf? should we even apply brdf before IOP inversion?*

What are the major challenges?

- Rayleigh-aerosol interaction in NUV, sensitivity to error in aerosol absorption*
- coupling of absorbing aerosols and CDOM in NUV*
- accurate correction for absorbing gases over NUV-SWIR (water vapor)*
- is solar irradiance knowledge sufficient (Thuillier 2003)?*

How can polarimeter measurements contribute to atmospheric correction, and what kind of polarimeter is required?

- is multi-angle required*
- what spectral bands?*
- co-registered to the radiometer swath?*
- co-registered to radiometer spatial sampling resolution?*

Can we reduce dependency on ancillary sources?

Where are the gaps (known issues, not being worked)?

Frouin: “atmospheric correction should”

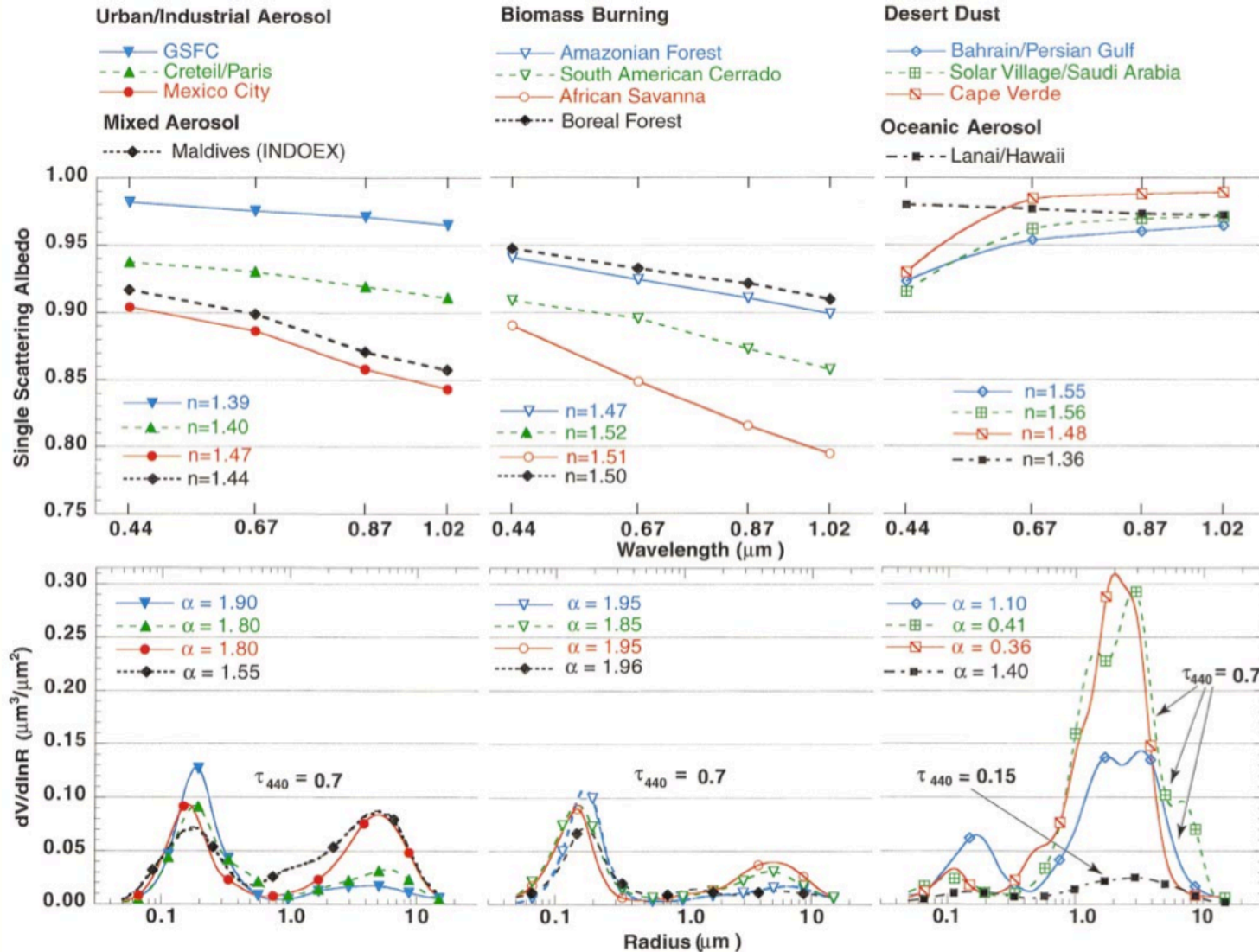
current NASA heritage algorithm

1. work in turbid, optically complex waters maybe, if NIR bio-optical model is valid
2. work in the presence of moderate sun glint maybe, if windspeed is accurate
3. work in the presence of semi-transparent clouds maybe, will be treated as aerosol
4. work in the presence of whitecaps yes, if windspeed is accurate
5. work when air mass is large how large? plane-parallel assumed
6. handle adjacency effects no, only instrument straylight, no atmospheric adjacency
– due to proximity of clouds, ice, land, etc.
7. handle situations of absorbing aerosols no, spectral dep in NIR is not unique
8. provide per-pixel uncertainties on Rrs no, not directly
9. be insensitive to radiometric calibration errors ?

and, work in open waters, in the presence of non-absorbing, low concentration aerosols, no adjacency effects, no whitecaps, no glint, no spherical effects yes, with uncertainty of order $1e-3 \text{ sr}^{-1}$ based on match-ups

back-up

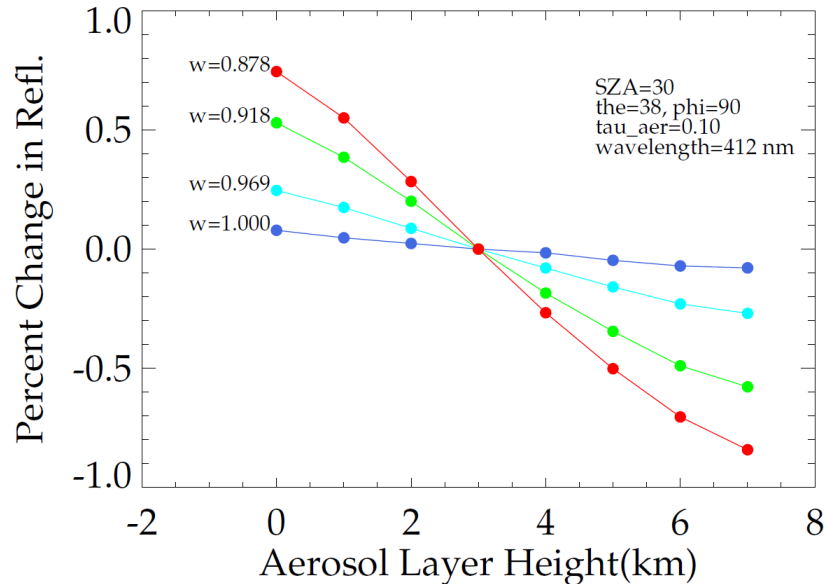
absorbing aerosols



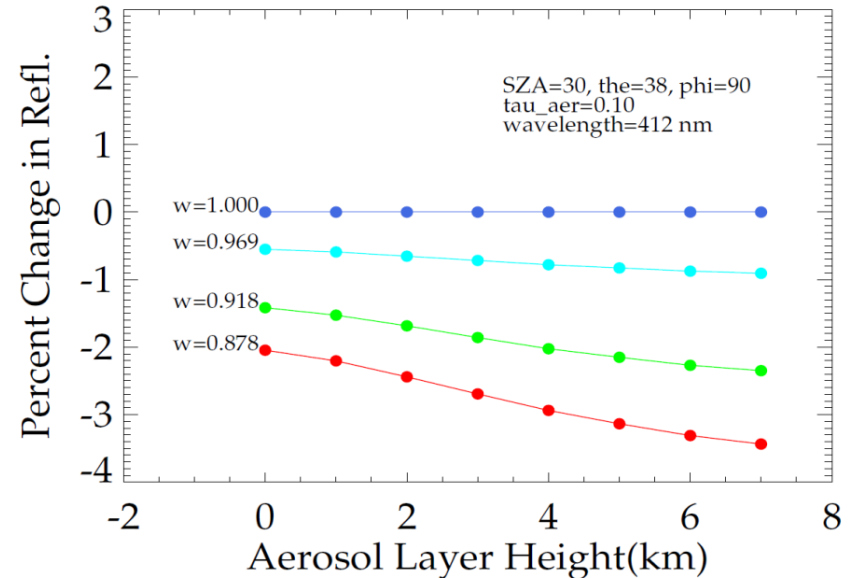
Dubovik, O., B. Holben, T. F. Eck, A. Smirnov, Y. J. Kaufman, M. D. King, D. Tanré, and I. Slutsker (2002), Variability of absorption and optical properties of key aerosol types observed in worldwide locations, *J. Atmos. Sci.*, 59, 590-608.

absorbing aerosols

effect of aerosol layer height on top-of-atmosphere reflectances



For dust ($\omega=0.878$) & $\tau_a=0.1$, a 1-km error in aerosol layer height corresponds to 0.3% difference in L_t . This translates into a 3% difference in L_w . The error increases with increasing τ_a .



For an aerosol layer at 3-km & $\tau_a=0.1$, a change from $\omega=0.878$ to $\omega=0.918$ corresponds to 1% difference in L_t . This translates into a 10% difference in L_w . The error increases with increasing τ_a .

